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1. Introduction

Simplicial algorithms have been developed on the n-dimensional unit or price simplex $S^n = \{p \in R_+^{n+1} \mid \Sigma_{j=1}^{n+1} p_j = 1\}$ in order to approximate economic equilibria in a pure exchange economy or to solve the nonlinear complementarity problem on S^n. The latter problem concerns the finding of a point p in S^n for which $z(p) \leqq 0$ where z is a continuous function from S^n to R^{n+1} satisfying (Walras' law) $p^T z(p) = 0$ for all p. In case of a pure exchange economy the function z is the excess demand function satisfying also $z_i(p) > 0$ if $p_i = 0$ (desirability condition). A simplicial algorithm subdivides S^n into n-dimensional simplices, or n-simplices, and searches for an n-simplex which yields an approximate solution. By refining the simplicial subdivision the accuracy of the approximation might be improved. A so-called simplicial variable dimension restart algorithm can start anywhere and generates a sequence of adjacent simplices of the subdivision of varying dimension until within a finite number of steps an approximate solution is found. In fact the algorithm traces a piecewise linear path from the starting point to the approximate solution, which is contained in the sequence of simplices generated by the algorithm. The piecewise linear path is followed by alternating replacement steps in the simplicial subdivision in order to move from one simplex to an adjacent one and by linear programming pivot steps in a system of n+2 linear equations in order to trace a linear piece of the path in a given simplex. The starting point, v, is left with one-dimensional simplices along a ray. In the so-called (n+1)-ray algorithm, due to van der Laan and Talman [5], see also [1], there is a ray leading to each of the n+1 vertices of S^n. The algorithm leaves v along the ray to the vertex e(i) of S^n if $z_i(v) = \max_h z_h(v)$, where e(i) is the i-th unit vector in R^{n+1}, i=1,...,n+1.

This algorithm can be considered as a (price) adjustment procedure having an attractive economic interpretation when applied to find an equilibrium in a pure exchange economy, see e.g. Zangwill and Garcia [12], van der Laan and Talman [6, 7]. In this way the algorithm is an always converging alternative for the classical Walras tatonnement process. Simplicial algorithms can also be used to find equilibria in an exchange economy with linear production technology, see e.g. Scarf [11]. In [11]

an upper semicontinuous mapping is constructed where the function value at a price is set equal to the production technology vector of the activity making the largest positive profit and to the excess demand vector if no activity makes (a positive) profit. Probably due to the discontinuity, in particular close to the equilibrium where all the activities in use make zero profit, the computational results look not very promising. A different approach to use simplicial algorithms for finding equilibria in a production economy is to construct a continuous function from S^n to S^n whose fixed points are equilibria and conversely. For example, see e.g. Kehoe [3], the function g defined by g(p) is the projection of the vector p+z(p) on the set of S^n where no activity makes any profit satisfies this property. In that case, however, at each iteration of the algorithm on S^n the projection of p+z(p) has to be computed by a linear complementarity problem (LCP) pivoting algorithm which takes typically at least n+1 additional linear programming steps per iteration. We remark that the function g was never introduced to find equilibria but only to show existence or to analyse unicity and stability issues.

In Mathiesen [8], see also Mathiesen [9], Mathiesen and Rutherford [10] and Eaves [2], equilibrium problems were solved by a sequence of linear complementarity problems (SLCP) where each LCP in the sequence has to be solved by a pivoting algorithm. Although the computational results reported are very exciting, there is not much theoretical understanding of SLCP. In such a scheme it is not clear in general whether all the LCP's in the sequence can be solved by a pivoting algorithm or even have a solution. Probably this is due to the fact that (a priori) a numeraire is chosen so that the LCP's are defined on the unbounded set R^n_+. However, by solving the induced LCP's directly on S^n without having a numeraire a pivoting algorithm always finds a solution, see e.g. [2]. The global convergence is still an unsolved issue for an SLCP scheme except in special cases. Since SLCP is a natural extension of Newton's method to include linear constraints, there is not much hope for such an iterative method to converge for an arbitrary excess demand function and activity matrix. Moreover, the economic interpretation of this method is not very clear.

In this paper we introduce a simplicial variable dimension algorithm on the set $S^n \times \mathbb{R}^m_+$ where m is the number of production technolo-

gies in the economy. The algorithms solves the problem of finding an equilibrium in the economy with linear production technology directly, i.e. the problem is not converted to an equivalent problem on S^n and the algorithm does not need to solve an LCP at each iteration. Moreover, the algorithm exploits the linearity of the problem in the activity variables and finds, barring degeneracy, within a finite number of iteration an approximating solution under the condition that something cannot be produced from nothing. Due to the linearity of the problem in the activity variables, the set $S^n \times R_+^n$ is not subdivided in (n+m)-simplices but in principle in (n+m)-cells being the product of an n-simplex of a simplicial subdivision of S^n and R_+^n. When m is equal to zero, that is when there is no production allowed in the economy, the algorithm reduces to the (n+1)-ray algorithm on S^n of Doup and Talman [1].

The steps of the algorithm can be considered as a global adjustment process to attain an equilibrium, similar to the one induced by the (n+1)-ray algorithm on S^n. In particular, the algorithm increases initially the price of the commodity with the largest netto excess demand or the level of activity with the largest (positive) profit. During the algorithm both the prices of the commodities with maximal (piecewise linear) netto excess demand and the activity levels of the production technologies with equal maximal profit are, relatively to the starting values, larger than all the other prices and activity levels. These are, again relatively to the starting values, all equal to each other. The algorithm terminates as soon as the profits of all the technologies with positive production level and the netto excess demands of the commodities with positive prices are equal to each other. According to Walras' law the algorithm has then found an approximating equilibrium. In fact the algorithm traces a piecewise linear path from the starting point to an approximating equilibrium. This path is followed by a sequence of adjacent cells of varying dimension of the underlying subdivision of $S^n \times R_+^m$. In each cell the linear piece of the path is followed by a linear programming pivot step in a system of n+1+m+1 linear equations. The movement from one cell of the subdivision to an adjacent one is again described by a replacement rule.

The paper is organized as follows. Section 2 describes the problem and defines the piecewise linear path of the algorithm. In section 3

the subdivision of $S^n \times R_+^m$ in cells which underlies the algorithm is given. Section 4 gives the steps to follow the piecewise linear path traced by the algorithm.

2. The path of the adjustment process

Consider an economy with linear production technology, initial endowments and demand functions. Let n+1 be the number of commodities and m the number of activities. Then

p = $(p_1,\ldots,p_{n+1})^T$ denotes the vector of prices
w = $(w_1,\ldots,w_{n+1})^T$ denotes the vector of endowments
$d(p)$ = $(d_1(p),\ldots,d_{n+1}(p))^T$ denotes the aggregate demand function
y = $(y_1,\ldots,y_m)^T$ denotes the vector of activity levels
A denotes the $m \times n$ technology matrix of input-output coefficients consistent with unit production, where the (i,j)th element $a_{ij} > 0$ ($a_{ij} < 0$) denotes an output (input).

We assume that the demand functions d are continuous functions from R_+^{n+1} into R^{n+1}.
Moreover we assume that the technology matrix A does not allow for production without input, that is $y \geqq 0$ and $Ay \geqq 0$ implies $y=0$.

Definition 2.1
A vector $(p^*,y^*) \in R_+^{n+1} \times R_+^m$ constitutes a general equilibrium if

$$A^T p^* \leqslant 0 \tag{2.1}$$
$$d(p^*) - w - Ay^* \leqslant 0 \tag{2.2}$$
$$p^{*T} Ay^* = 0 \tag{2.3}$$
$$p^{*T}(w+Ay^*-d(p^*)) = 0. \tag{2.4}$$

Equation (2.1) requires that no activity earns a positive profit, (2.2) that no commodity is in excess demand. Equation (2.3) reflects that an activity level is zero if the profit is negative and an operated activity has profit zero. Finally (2.4) means that the commodity in excess supply has zero price and a positive price implies market clearance. We assume that the demand functions are derived from individual household utility maximalization. When the demands satisfy each individual household's budget and there is nonsatiation then $p^T d(p) = p^T w$.

We call $z(p) = d(p) - w$ the excess demand function (e.d.f), satisfying $p^T z(p) = 0$ for all $p \in R_+^{n+1}$ (Walras' law), and $z_j(p) - (Ay)_j$ the netto excess demand of commodity j. Moreover the e.d.f.'s are homogeneous of degree zero so that we can normalize prices p to the n-dimensional unit simplex $S^n = \{p \in R_+^{n+1} \mid \Sigma_{j=1}^{n+1} p_j = 1\}$. Under the conditions mentioned above there always exists a general equilibrium $(p^*, y^*) \in S^n \times R_+^m$.
In order to find such an equilibrium vector we propose to follow the path of points (p,y) in $S^n \times R_+^m$ from an arbitrary starting point $(\bar{p},\bar{y})$ satisfying for some β and min, $0 < \min < 1$,

$$\begin{aligned} p_j &= \bar{p}_j \min && \text{if } z_j(p) - (Ay)_j < \beta \\ p_j &> \bar{p}_j \min && \text{if } z_j(p) - (Ay)_j = \beta \\ y_i &= \bar{y}_i \min && \text{if } (p^T A)_i < \beta \\ y_i &> \bar{y}_i \min && \text{if } (p^T A)_i = \beta. \end{aligned} \tag{2.5}$$

Note that $\beta = \max \{\max_j z_j(p) - (Ay)_j, \max_i (p^T A)_i\} > 0$ and that

$$\min = \min \{\min_{\bar{p}_j > 0} \frac{p_j}{\bar{p}_j}, \min_{\bar{y}_i > 0} \frac{y_i}{\bar{y}_i}\}.$$

The starting point $(\bar{p},\bar{y})$ satisfies (2.5) with min = 1. If z is continuously differentiable then under some non-degeneracy conditions system (2.5) implies the existence of a piecewise smooth path connecting $(\bar{p},\bar{y})$ and an equilibrium vector (p^*,y^*), in the same way as has been derived for a pure exchange economy in van der Laan en Talman [7]. The piecewise smooth path leaves $(\bar{p},\bar{y})$ by either increasing the price of the commodity with the largest netto excess demand, if larger than the largest profit $\max_i (p^T A)_i$, or increasing the activity level of the production technology with the largest profit if the latter is larger than the largest netto excess demand. In the first case all the other prices and all the activity levels are decreased proportionally in order to keep the sum of the prices equal to one and since the profit of each activity is smaller than the largest netto excess demand. In the second case the other activity levels and all the prices are kept equal to their initial value in order to keep the sum of the prices equal to one.

In general on the path the prices of the commodities having largest netto excess demand and the activity levels of the technologies with the same largest profits are, relatively to the starting values, larger than all the other prices and activity levels, which are all the same proportion (min) smaller than the initial values. As soon as the netto excess demand $z_h(p) - (Ay)_h$ of commodity h becomes equal to β for some h for which $p_h = \min \bar{p}_h$ then the price p_h is increased away from $\bar{p}_h \min$ and $z_h(p) - (Ay)_h$ is kept equal to β. Similarly if the profit $(p^TA)_h$ of technology h becomes equal to β for some h for which $y_h = \bar{y}_h \min$ then the level y_h is increased from $\bar{y}_h \min$ and the profit $(p^TA)_h$ is kept equal to β. On the other hand if the price p_k of commodity k becomes equal to $\bar{p}_k \min$ for some k for which $z_k(p) - (Ay)_k = \beta$ then p_k is kept equal to $\bar{p}_k \min$ and the netto excess demand $z_k(p) - (Ay)_k$ is decreased from β. Finally if the level y_k of technology k becomes equal to $\bar{y}_k \min$ for some k for which $(p^TA)_k = \beta$ then y_k is kept equal to $\bar{y}_k \min$ and the profit $(p^TA)_k$ is decreased from β.

The path of points (p,y) from $(\bar{p},\bar{y})$ satisfying (2.5) terminates with a point (p^*,y^*) for which

$$\begin{aligned} &z_j(p^*) - (Ay^*)_j \leq \beta^* \text{ for all } j=1,\ldots,n+1 \\ &\text{with } z_j(p^*) - (Ay^*)_j < \beta^* \text{ implies } p_j^* = 0, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &(p^{*T}A)_i \leq \beta^* \text{ for all } i=1,\ldots,m \\ &\text{with } (p^{*T}A)_i < \beta^* \text{ implies } y_i^* = 0. \end{aligned} \tag{2.7}$$

Multiplying (2.6) by p_j^* and adding up over all j yields

$$p^{*T}z(p^*) - p^{*T}Ay^* = \beta^* \sum_{i=1}^{n+1} p_j^* .$$

Since $p^{*T}z(p^*) = 0$ and $\Sigma_{i=1}^{n+1} p_j^* = 1$ it follows that $p^{*T}Ay^* = -\beta^*$. Multiplying (2.7) by y_i^* and adding up over all i yields $p^{*T}Ay^* = \beta^* \Sigma_{i=1}^{m} y_i^*$ Consequently $(1 + \Sigma_{i=1}^{m} y_i^*)\beta^* = 0$, so that since $\Sigma_{i=1}^{m} y_i^* > 0$, $\beta^* = 0$. Hence all the netto excess demands are less than or equal to

zero, no activity makes profit and a technology with positive activity level has zero profit.

The existence of the piecewise smooth path P from $(\bar{p},\bar{y})$ to (p^*,y^*) can be proved by introducing a primal-dual pair of subdivided manifolds on which a mapping is defined whose zero points yield the solutions to (2.5) (see e.g. Kojima and Yamamoto [4]).

In the rest of this paper we describe an algorithm which follows the piecewise smooth path P approximately by a piecewise linear path $\bar{P}$. The path $\bar{P}$ is obtained from system (2.5) by taking the piecewise linear approximation $\bar{z}$ to z with respect to a specific subdivision of $S^n \times R^m_+$. The subdivision is described in the next section and the steps of the algorithm in section 4.

3. The subdivision of $S^n \times R^m_+$

The subdivision of $S^n \times R^m_+$ in (m+n)-dimensional cells is based on the V-triangulation of S^n introduced in Doup and Talman [1]. In principle the subdivision consists of cells which are the Carthesian product of the n-simplices of the V-triangulation of S^n and of R^m_+. Since the system (2.5) is linear in the variables y_i, i=1,...,m, the algorithm does not need a simplicial subdivision of R^m_+. Let $(\bar{p},\bar{y})$ be the starting point of the algorithm.

For $T \subsetneq I_{n+1+m} = \{1,2,\ldots,n+1+m\}$ we define $A(T) = \emptyset$ if both $\bar{p}_j = 0$ for all $j \notin T \cap I_{n+1}$ and $\bar{y}_i = 0$ for all i with $n+1+i \notin T$, and otherwise

$$A(T) = \{(p,y) \in S^n \times R^m_+ \mid p_h \geq \bar{p}_h \min,\ h \in T,\ p_h = \bar{p}_h \min,\ h \notin T,\ y_i \geq \bar{y}_i \min,\ n+1+i \in T,\ y_i = \bar{y}_i \min,\ n+1+i \notin T,\ 0 \leq \min \leq 1\}.$$

The dimension of a nonempty set A(T) is equal to $t = |T|$. Notice that if $I_{n+1} \cap T = \emptyset$ then min must be equal to 1 so that

$$A(T) = \{(\bar{p},y) \in S^n \times R^m_+ \mid y_i \geq \bar{y}_i,\ n+1+i \in T, y_i = \bar{y}_i, n+1+i \notin T\}.$$

The boundary of A(T) is equal to

$$\text{Bd } A(T) = \bigcup_{h \in T} A(T\setminus\{h\}) \cup \{(p,y) \in A(T)\ p_j = 0,\ j \notin T \cap I_{n+1},\ y_i = 0,\ n+1+i \notin T\}.$$

The sets A(T) reflect the left part of (2.5) and are illustrated in figure 3.1 for n=m=1 and in figure 3.2 for n=2, m=1.

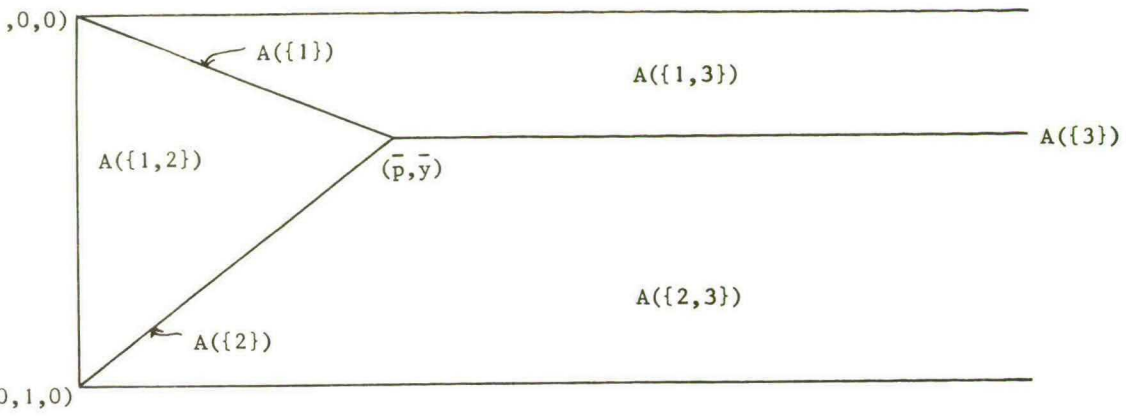

Figure 3.1. The sets A(T) for n=m=1

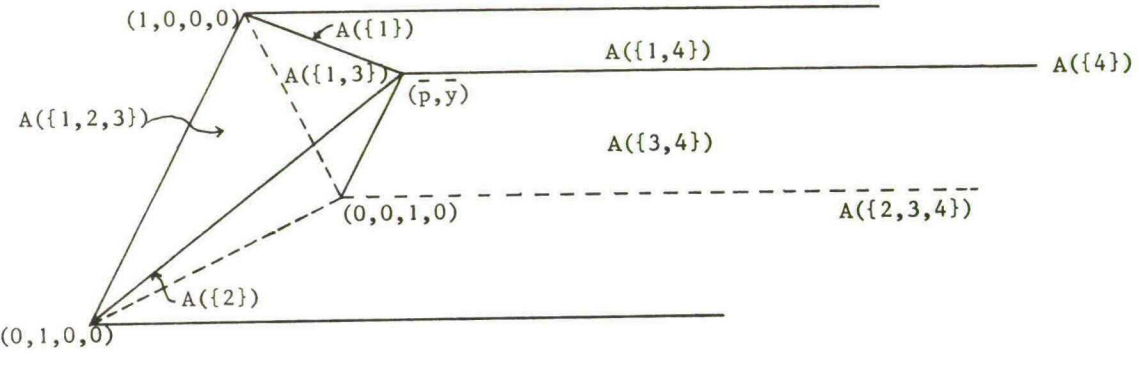

Figure 3.2. Some sets A(T) if n=2 and m=1

Let the intersection of a subset T of I_{n+1+m} with I_{n+1} be denoted by T_1 and its intersection with $\{n+1+1,\ldots,n+1+m\}$ by T_2. Let e(k) be the unit vector in $\mathbb{R}^{n+1+m}$ with $e_i(k) = 1$ if $i=k$ and $e_i(k) = 0$ if $i \neq k$. Then the t-dimensional set A(T) is the closure of the convex hull of $(\bar{p},\bar{y})$, e(j), $j \in T_1$, and the rays $\{(\bar{p},\bar{y}) + \lambda\, e(n+1+i),\ \lambda > 0\}$, $n+1+i \in T_2$. In particular $A(\{j\})$ is co $\{e(j),(\bar{p},\bar{y})\}$ if $j \in I_{n+1}$ and $A(\{n+1+i\}) = \{(\bar{p},\bar{y}) + \lambda\, e(n+1+i),\ \lambda > 0\}$ if $i \in I_m$. So there are n+1+m one-dimensional sets A(T) with $|T| = 1$ if $(\bar{p},\bar{y}) \neq e(j)$, $j \in I_{n+1}$, and there are just n+m if $(\bar{p},\bar{y}) = e(j)$ for some $j \in I_{n+1}$. Along one of these one-dimensional sets the algorithm will leave $(\bar{p},\bar{y})$. We remark that in the sequel

the element (p,y) in $S^n \times R_+^m$ will be used to denote the $(n+1+m)$-vector $(p^T,y^T)^T$ in R_+^{n+1+m}.

Analogue to the V-triangulation of S^n we divide a nonempty set $A(T)$ into the regions $A(\gamma(T_1),T_2)$ for any permutation $\gamma(T_1) = (\gamma(1),\ldots,\gamma(t_1))$ of the t_1 elements of T_1. For $K \neq I_{n+1}$ let $p(K)$ be the relative projection of $\bar{p}$ on the face $S^n(K) = \{p \in S^n \mid p_j = 0,\ j \notin K\}$, of S^n i.e., $p(K) = \bar{p}$ if $K = \emptyset$ and $p_j(K) = 0$ for all $j \notin K$ if $K \neq \emptyset$. Further in the latter case, if $\sum_{k\in K} \bar{p}_k < 1$

$$p_j(K) = \frac{\bar{p}_j(1+|K^0|)}{\sum_{h\in K} \bar{p}_h + |K^0|}, \qquad j \in K \setminus K^0$$

and

$$p_j(K) = \frac{1 - \sum_{h\in K} \bar{p}_h}{\sum_{h\in K} \bar{p}_h + |K^0|}, \qquad j \in K^0,$$

and if $\sum_{k\in K} \bar{p}_k = 1$

$$p_j(K) = \frac{\bar{p}_j}{1 + |K^0|}, \qquad j \in K \setminus K^0$$

and

$$p_j(K) = \frac{1}{1 + |K^0|}, \qquad j \in K^0$$

where $K^0 = \{k \in K \mid \bar{p}_k = 0\}$.

Furthermore let

$$q(\gamma(1)) = \begin{pmatrix} p(\{\gamma(1)\}) - p(\emptyset) \\ -\bar{y} \end{pmatrix} = e(\gamma(1)) - \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix}$$

$$q(\gamma(i)) = \begin{pmatrix} p(\{\gamma(1),\ldots,\gamma(i)\}) - p(\{\gamma(1),\ldots,\gamma(i-1)\}) \\ 0 \end{pmatrix} \quad i=2,\ldots,t_1$$

and let $q(0) = \begin{pmatrix} \bar{p} \\ 0 \end{pmatrix} - \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -\bar{y} \end{pmatrix}$.

Definition 3.1

Suppose A(T) is nonempty. If T_1 is a proper subset of I_{n+1} such that there is a $j \notin T_1$ with $\bar{p}_j > 0$, then

$$A(\gamma(T_1),T_2) = \{(p,y) \in S^n \times R_+^m \mid$$

$$\begin{pmatrix} p \\ y \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} + \sum_{j=1}^{t_1} \alpha_j \, q(\gamma(j)) + \sum_{n+1+i \in T_2} \nu_{n+1+i} e(n+1+i) \text{ with}$$

$$0 < \alpha_{t_1} < \ldots < \alpha_1 < 1, \ \nu_{n+1+i} > 0, \ n+1+i \in T_2\}. \tag{3.1}$$

If $T_1 = I_{n+1}$ or $j \notin T_1$ implies $\bar{p}_j = 0$, and if T is a proper subset of I_{n+1+m} then

$$A(\gamma(T_1),T_2) = \{(p,y) \in S^n \times R_+^m \mid$$

$$\begin{pmatrix} p \\ y \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} + \sum_{j=1}^{t_1-1} \alpha_j \, q(\gamma(j)) + \alpha_0 q(0) + \sum_{n+1+i \in T_2} \nu_{n+1+i} e(n+1+i)$$

$$\text{with } 0 < \alpha_{t_1-1} < \ldots < \alpha_2 < \alpha_1 < 1-\alpha_0 < 1 \text{ and}$$

$$\nu_{n+1+i} > 0, \ n+1+i \in T_2\}. \tag{3.2}$$

Otherwise $A(\gamma(T_1),T_2)$ is defined to be empty.
The union of $A(\gamma(T_1),T_2)$ over all permutations $\gamma(T_1)$ is clearly A(T) with $T = T_1 \cup T_2$, and each nonempty $A(\gamma(T_1),T_2)$ has dimension $t = t_1+t_2$.

If $\bar{p}_j > 0$ for all $j \in T_1$ then $A(\gamma(T_1),T_2)$ can be written as follows.

Lemma 3.2

Suppose $\bar{p}_j > 0$ for all $j \in T_1$ and $T_1 \cup T_2$ is such that $A(\gamma(T_1),T_2) \neq \emptyset$ then

$$A(\gamma(T_1),T_2) = \{(p,y) \in S^n \times R_+^m \mid \frac{p_{\gamma(1)}}{\bar{p}_{\gamma(1)}} \geq \ldots \geq \frac{p_{\gamma(t_1)}}{\bar{p}_{\gamma(t_1)}} \geq \min,$$

$$p_h = \bar{p}_h \min \quad h \notin T_1,$$

$$y_i \geq \bar{y}_i \min \quad n+1+i \in T_2,$$

$$y_i = \bar{y}_i \min \quad n+1+i \notin T_2,$$

$$0 \leq \min \leq 1\}. \qquad (3.3)$$

Notice that min in (3.3) is equal to $1-\alpha_1$ in (3.1) and is equal to $1-\alpha_1-\alpha_0$ in (3.2).

The sets $A(\gamma(T_1),T_2)$ are illustrated in figure 3.3 for n=m=1 and some are illustrated in figure 3.4 for n=2 and m=1.

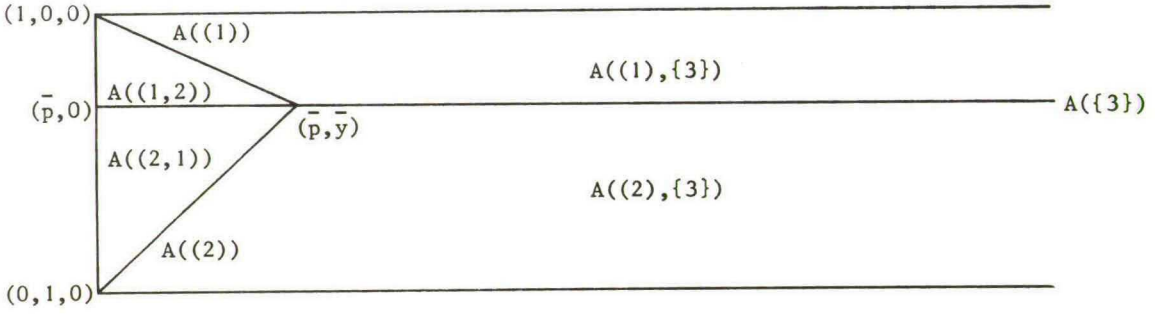

Figure 3.3. The sets $A(\gamma(T_1),T_2)$ for n=m=1

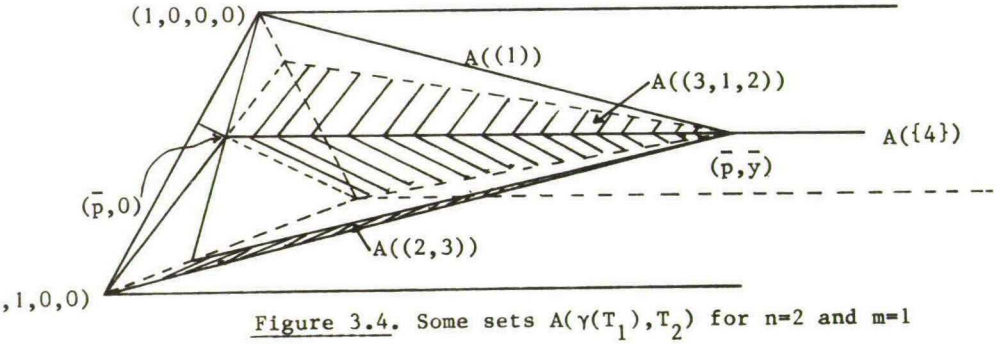

Figure 3.4. Some sets $A(\gamma(T_1),T_2)$ for n=2 and m=1

Let $A'(\gamma(T_1))$ be the projection of $A(\gamma(T_1),T_2)$ on S^n i.e. if there is a $j \notin T_1$ with $\bar{p}_j > 0$ then

$$A'(\gamma(T_1)) = \{p \in S^n \mid p = \bar{p} + \sum_{j=1}^{t_1} \alpha_j q'(\gamma(j)),$$

$$0 < \alpha_{t_1} < \ldots < \alpha_1 < 1 \},$$

and if $j \notin T_1$ implies $\bar{p}_j = 0$ then

$$A'(\gamma(T_1)) = \emptyset \qquad \text{if } \bar{p}_{\gamma(t_1)} = 0$$

and if $\bar{p}_{\gamma(t_1)} > 0$

$$A'(\gamma(T_1)) = \{p \in S^n \mid p = \bar{p} + \sum_{j=1}^{t_1-1} \alpha_j q'(\gamma(j),$$

$$0 < \alpha_{t_1-1} < \ldots < \alpha_1 < 1\},$$

where $q'(\gamma(j)) \in R^{n+1}$, $q'_h(\gamma(j)) = q_h(\gamma(j)) \quad h \in I_{n+1}$.
Notice that if in the latter case $A'(\gamma(T_1))$ is nonempty then $A'((\gamma(1), \ldots,\gamma(t_1))) = A'((\gamma(1),\ldots,\gamma(t_1-1)))$.

The t-cell σ of the subdivision of a nonempty $A(\gamma(T_1),T_2)$ is the intersection of $\sigma' \times R_+^m$ with $A(\gamma(T_1),T_2)$ where σ' is a t_1-simplex in $A'(\gamma(T_1))$ of the V-triangulation of S^n with certain grid size d^{-1}, $d \in \mathbb{N}$. Such a t_1-simplex σ' can be described by $\sigma'(p^1, \pi(T_1))$ with vertices $p^1,\ldots,p^{t_1+1}$ such that

(i) $p^1 = \bar{p} + \sum_{j\ T_1} a(j)d^{-1}q'(j)$ for non negative integers $a(j)$ with

$$0 < a(\gamma(t_1)) < \ldots < a(\gamma(1)) < d-1$$

(ii) $\pi(T_1) = (\pi(1),\ldots,\pi(t_1))$ is a permutation of the t_1 elements of T_1 such that for all $j=2,\ldots,t_1$: $k' > k$ if $a(\gamma(j-1)) = a(\gamma(j))$, $\pi_k = \gamma(j-1)$ and $\pi_{k'} = \gamma(j)$

(iii) $p^{i+1} = p^i + d^{-1}q'(\pi(i))$ $i=1,\dots,t_1$.

If $T_1 = I_{n+1}$ or $j \notin T_1$ implies $\bar{p}_j = 0$ then σ' in $A'(\gamma(T_1))$ is a (t_1-1)-simplex $\sigma'(p^1, \pi(T_1))$ with $\pi(t_1) = \gamma(t_1)$, $a(\gamma(t_1)) = 0$ and vertices $p^1,\dots,p^{t_1}$ as above in (iii) for $i=1,\dots,t_1-1$.

First we describe the subdivision of $A(\gamma(T_1),T_2)$ if $T_2 = \emptyset$ and there is a $j \notin T_1$ for which $\bar{p}_j > 0$. This subdivision is a simplicial subdivision in t_1-simplices $\sigma((p^1,y^1), \pi(T_1), \emptyset)$ with vertices $(p^1,y^1),\dots,(p^{t_1+1}, y^{t_1+1})$ such that

(i) $\begin{pmatrix} p^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} + \sum_{j\in T_1} a(j)d^{-1}\, q(j)$ for nonnegative integers with

$0 \leq a(\gamma(t_1)) \leq \dots \leq a(\gamma(1)) \leq d-1$

(ii) $\pi(T_1) = (\pi(1),\dots,\pi(t_1))$ is a permutation of the elements of T_1 such that for all $j=2,\dots,t_1$: $k' > k$ if $a(\gamma(j-1)) = a(\gamma(j))$, $\pi_k = \gamma(j-1)$ and $\pi_{k'} = \gamma(j)$

(iii) $\begin{pmatrix} p^{i+1} \\ y^{i+1} \end{pmatrix} = \begin{pmatrix} p^i \\ y^i \end{pmatrix} + d^{-1}q(\pi(i))$ $i=1,\dots,t_1$.

Notice that $\sigma'(p^1, \pi(T_1))$ is the projection of $\sigma((p^1,y^1), \pi(T_1), \emptyset)$ on S^n. If $j \notin T_1$ implies $\bar{p}_j = 0$ or $T_2 \neq \emptyset$ then the subdivision of a non-empty $A(\gamma(T_1),T_2)$ consists of t-cells. In case there is a $j \notin T_1$ with $\bar{p}_j > 0$ such a cell is described by

$$\sigma((p^1,y^1), \pi(T_1),T_2) = \{(p,y) \in S^n \times R_+^m \mid$$

$$\begin{pmatrix} p \\ y \end{pmatrix} = \sum_{j=1}^{t_1+1} \lambda_j \begin{pmatrix} p^j \\ y^j \end{pmatrix} + \sum_{n+1+i\in T_2} \nu_{n+1+i} e(n+1+i),\ \lambda_j > 0 \quad j \in I_{t_1+1},$$

$$\sum_{j=1}^{t_1+1} \lambda_j = 1,\ \nu_{n+1+i} > 0 \quad n+1+i \in T_2, \text{ where}$$

$(p^1,y^1),\dots,(p^{t_1+1}, y^{t_1+1})$ are the vertices of the t_1-simplex $\sigma((p^1,y^1), \pi(T_1), \emptyset)\}$. (3.4)

Remark that we may allow T_2 to be empty. In case $T_1 = I_{n+1}$ or $j \notin T_1$ implies $\bar{p}_j = 0$ a t-cell of a nonempty $A(\gamma(T_1),T_2)$ is described by

$$\sigma((p^1,y^1), \pi(T_1),T_2) = \{(p,y) \in S^n \times R_+^m |$$

$$\begin{pmatrix} p \\ y \end{pmatrix} = \sum_{j=1}^{t_1} \lambda_j \begin{pmatrix} p^j \\ y^j \end{pmatrix} + \alpha_0 \begin{pmatrix} 0 \\ -\bar{y} \end{pmatrix} + \sum_{n+1+i \in T_2} \nu_{n+1+i} e(n+1+i),$$

$$\lambda_j > 0 \quad j \in I_{t_1}, \quad \sum_{j \in I_{t_1}} \lambda_j = 1, \quad \nu_{n+1+i} > 0 \quad n+1+i \in T_2,$$

$$0 < \alpha_0 < \frac{p_{\gamma(t_1)}}{\bar{p}_{\gamma(t_1)}}, \text{ where } (p^1,y^1),\ldots,(p^{t_1},y^{t_1})$$

are the vertices of $\sigma((p^1,y^1), (\pi(1),\ldots,\pi(t_1-1)), \emptyset)\}$. (3.5)

Recall that in this case $\bar{p}_{\gamma(t_1)} > 0$.

The union of the subdivisions of $A(\gamma(T_1),T_2)$ over all permutations yields a subdivision of A(T) in t-cells. The union of these subdivisions of A(T) over all feasible T constitutes a subdivision of $S^n \times R_+^m$ with grid size d^{-1}. For n=m=1 and d=3 the subdivision is illustrated in figures 3.5 and 3.6.

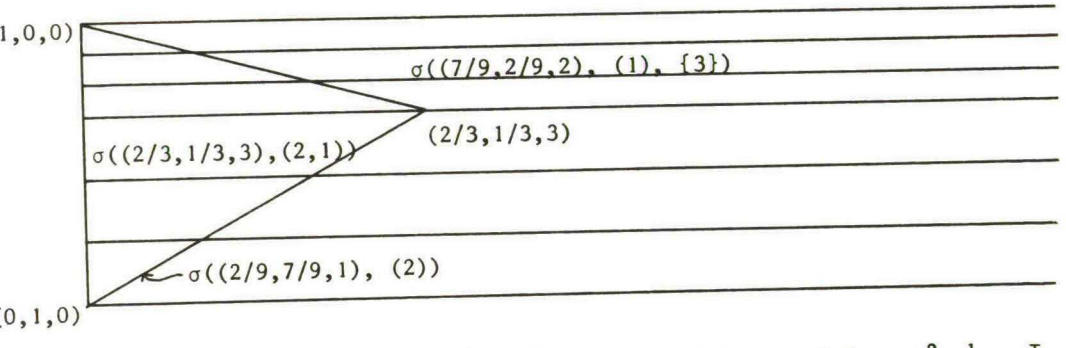

Figure 3.5. Subdivision of $S^n \times R_+^m$ if n=m=1, d=3 and $(\bar{p},\bar{y}) = (\frac{2}{3}, \frac{1}{3}, 3)^T$

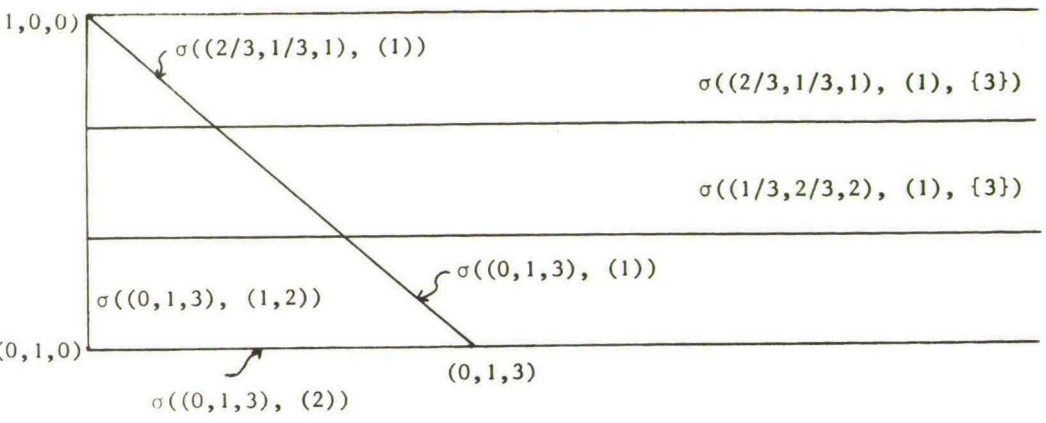

Figure 3.6. Subdivision of $S^n \times R_+^m$ if n=m=1, d=3 and $(\bar{p},\bar{y}) = (0, 1, 3)^T$

4. The algorithm

As mentioned in section 2 the algorithm traces the piecewise linear path $\bar{P}$ from $(\bar{p},\bar{y})$ of points (p,y) satisfying

$$\begin{aligned} p_j &= \bar{p}_j \min && \text{if } \bar{z}_j(p) - (Ay)_j < \beta \\ p_j &\geq \bar{p}_j \min && \text{if } \bar{z}_j(p) - (Ay)_j = \beta \\ y_i &= \bar{y}_i \min && \text{if } (p^T A)_i < \beta \\ y_i &\geq \bar{y}_i \min && \text{if } (p^T A)_i = \beta \end{aligned} \tag{4.1}$$

with $\bar{z}$ the piecewise linear approximation to z with respect to the subdivision defined in section 3. The path $\bar{P}$ is followed by the algorithm by a sequence of adjacent cells of varying dimension. More precisely the algorithm makes alternating replacement steps in order to move from one cell to an adjacent cell and linear programming pivot steps in order to follow a linear piece of $\bar{P}$ in a cell. The linear programming pivot steps are made in a system of linear equations corresponding to the right part of system (4.1). We call the t-cell containing the point (p,y) in $A(T)$ satisfying the right part of (4.1) T-complete. Let A_i be the i-th column of A and let e be the (n+1+m)-vector of ones.

Definition 4.1

A t-cell $\sigma((p^1,y^1), \pi(T_1),T_2)$ in a nonempty $A(\gamma(T_1),T_2)$ is T-complete if

(i) in case there is $j \notin T_1$ with $\bar{p}_j > 0$ the system

$$\sum_{j=1}^{t_1+1} \lambda_j \begin{pmatrix} z(p^j) - Ay^j \\ A^T p^j \\ 1 \end{pmatrix} + \sum_{n+1+i \in T_2} \nu_{n+1+i} \begin{pmatrix} -A_i \\ 0 \\ 0 \end{pmatrix} + \sum_{h \notin T} \mu_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.2}$$

has a solution $\lambda_j \geq 0 \quad j \in I_{t_1+1}$, $\nu_{n+1+i} \geq 0 \quad n+1+i \in T_2$, $\mu_h \geq 0 \quad h \notin T$ and $\beta \in R$

(ii) in case $T_1 = I_{n+1}$ or $j \notin T$ implies $\bar{p}_j = 0$ the system

$$\sum_{j=1}^{t_1} \lambda_j \begin{pmatrix} z(p^j) - Ay^j \\ A\, p^j_1 \end{pmatrix} + \sum_{n+1+i \in T_2} \nu_{n+1+i} \begin{pmatrix} -A_i \\ 0 \\ 0 \end{pmatrix} + \sum_{h \notin T} \mu_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} + \alpha_0 \begin{pmatrix} A\bar{y} \\ 0 \\ 0 \end{pmatrix}$$

$$- \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (4.3)$$

has a solution $\lambda_j > 0 \quad j \in I_{t_1}$, $\nu_{n+1+i} > 0 \quad n+1+i \in T_2$, $\mu_h > 0 \quad h \notin T$, $\beta \in R$, $0 \leq \alpha_0 \leq (d - a(\gamma(1)) - \sum_{h=i}^{t_1} \lambda_h)/d$ where i is such that $\pi(i) = \gamma(1)$.

The solutions are denoted by $(\lambda, \nu, \mu, \beta) \in R^{n+m+3}$ in case (i) and $(\lambda, \nu, \mu, \alpha_0, \beta) \in R^{n+m+3}$ in case (ii). Notice that $\bar{z}(p^j) = z(p^j)$ and that in case (ii) $(d - a(\gamma(1)) - \sum_{h=i}^{t_1} \lambda_h)/d = (\sum_{j=1}^{t_1} \lambda_j p^j_{\gamma(t_1)})/\bar{p}_{\gamma(t_1)}$.

Lemma 4.2

Let (p,y) satisfy (4.1), then there exists a t-cell σ in $A(T)$ for some T, containing (p,y), which is T-complete. When σ is T-complete in $A(T)$ the point

$$\begin{pmatrix} p \\ y \end{pmatrix} = \sum_{j=1}^{t_1+1} \lambda_j \begin{pmatrix} p^j \\ y^j \end{pmatrix} + \sum_{n+1+i \in T_2} \nu_{n+1+i}\, e(n+1+i)$$

in case $\exists j \notin T_1$ with $\bar{p}_j > 0$ and

$$\begin{pmatrix} p \\ y \end{pmatrix} = \sum_{j=1}^{t_1} \lambda_j \begin{pmatrix} p^j \\ y^j \end{pmatrix} + \sum_{n+1+i \in T_2} \nu_{n+1+i}\, e(n+1+i) - \alpha_0 \begin{pmatrix} 0 \\ \bar{y} \end{pmatrix}$$

in case $T_1 = I_{n+1}$ or $j \notin T_1$ implies $\bar{p}_j = 0$ lies in σ and satisfies (4.1).

Proof. Let (p,y) satisfy (4.1). Then we define

$$T_1 = \{j \in I_{n+1} \mid \bar{z}_j(p) - (Ay)_j = \beta\}$$

and

$$T_2 = \{n+1+i \mid i \in I_m, (p^T A)_i = \beta\}.$$

Clearly (p,y) lies in $A(T)$ with $T = T_1 \cup T_2$. So there are $\gamma(T_1)$, $\pi(T_1)$ and (p^1,y^1) such that $(p,y) \in \sigma((p^1,y^1), \pi(T_1), T_2)$ with σ a t-cell in $A(\gamma(T_1), T_2)$. According to (3.4) or (3.5) σ is T-complete. The other part of the proof is trivial. □

Non-degeneracy assumption 4.3

At most one variable at a solution of (4.2) or (4.3) is equal to zero.

The non-degeneracy assumption implies that if the system (4.2) or (4.3) has a solution, then there is either a line segment or a half ray of solutions. At an endpoint of a line segment or half ray exactly one variable is equal to zero or α_0 is equal to the upperbound in (4.3).

Theorem 4.4

Neither system (4.2) nor system (4.3) has a half ray of solutions if T is nonempty.

Proof. Suppose that $(\lambda^0,\nu^0,\mu^0,\beta^0) + \alpha(\tilde{\lambda},\tilde{\nu},\tilde{\mu},\tilde{\beta})$ is a solution to (4.2) for all $\alpha \geq 0$.

Because of the last equation of (4.2) $\tilde{\lambda} = 0$. Moreover $\tilde{\mu} \geq 0$ and $\tilde{\nu} \geq 0$.

Let $p^0 = \sum_{j=1}^{t_1+1} \lambda_j^0 p^j$ and $y^0 = \sum_{j=1}^{t_1+1} \lambda_j^0 y^j$. Suppose $T_2 = \emptyset$ then (4.2) yields

$$\bar{z}(p^0) - Ay^0 + \sum_{h \notin T_1} (\mu_h^0 + \alpha\tilde{\mu}_h)e_1(h) - (\beta^0 + \alpha\tilde{\beta})e_1 = 0 \qquad (4.4)$$

where $e_1(h)$ and $e_1 \in \mathbb{R}^{n+1}$. Since T is nonempty the set $T_1 \neq \emptyset$. Now subtract (4.4) with $\alpha = 0$ from (4.4) to obtain

$$\sum_{h \notin T_1} \tilde{\mu}_h e_1(h) - \tilde{\beta}e_1 = 0.$$

The k-th equation with $k \in T_1$ yields $\tilde{\beta} = 0$ and therefore $\tilde{\mu}_h = 0$ for all $h \in I_{n+1} \setminus T$. Since $T_2 = \emptyset$ also $\tilde{\mu}_{n+1+i} = 0$ for all $i \in I_m$ and there is no ray.

Suppose $T_2 \neq \emptyset$ then system (4.2) yields

$$A^T p^0 + \sum_{\substack{h \notin T_2 \\ h-n-1 \in I_m}} (\mu_h^0 + \alpha\tilde{\mu}_h)\, e_2(h) - (\beta^0 + \alpha\tilde{\beta})\, e_2 = 0 \qquad (4.5)$$

where $e_2(h)$ and $e_2 \in \mathbb{R}^m$. Now subtract (4.5) with $\alpha = 0$ from (4.5) to obtain

$$\sum_{\substack{h \notin T_2 \\ h-n-1 \in I_m}} \tilde{\mu}_h\, e_2(h) - \tilde{\beta}\, e_2 = 0.$$

Hence $\tilde{\beta} = 0$ and also $\tilde{\mu}_h = 0$ for $h \notin T_2$, $h-n-1 \in I_m$. Now (4.2) yields

$$\bar{z}(p^0) - Ay^0 + \sum_{n+1+i \in T_2} (\nu_{n+1+i}^0 + \tilde{\nu}_{n+1+i}\alpha)(-A_i) + \sum_{\substack{h \notin T_1 \\ h \in I_{n+1}}} (\mu_h^0 + \alpha\tilde{\mu}_h) e_1(h) - \beta^0 e_1 = 0 \qquad (4.6)$$

Again subtracting (4.6) for $\alpha = 0$ from (4.6) for $\alpha = 1$ yields

$$\sum_{n+1+i \in T_2} \tilde{\nu}_{n+1+i}(-A_i) + \sum_{\substack{h \notin T_1 \\ h \in I_{n+1}}} \tilde{\mu}_h e_1(h) = 0. \qquad (4.7)$$

Let $\tilde{\tilde{\nu}}$ be the m-vector defined by

$$\tilde{\tilde{\nu}}_i = \begin{cases} \tilde{\nu}_{n+1+i} & \text{for } n+1+i \in T_2 \\ 0 & \text{for } n+1+i \notin T_2. \end{cases}$$

Clearly $\tilde{\tilde{\nu}} \geq 0$. Equation (4.7) yields $A\tilde{\tilde{\nu}} \geq 0$ and hence $\tilde{\tilde{\nu}} = 0$, since there can be no production without input. Consequently $\tilde{\mu}_h = 0$ for all $h \notin T_1$, $h \in I_{n+1}$, so that there can be no ray.

Analoguously we can prove the theorem for system (4.3). Notice that α_0 is bounded between 0 and $(d-a_1)/d$ so that $\tilde{\alpha}_0 = 0$. □

According to lemma 4.2 a T-complete cell $\sigma((p^1,y^1), \pi(T_1),T_2)$ in A(T) contains a line segment of points (p,y) satisfying (4.1). At an endpoint of such a line segment just one of the variables λ_j, ν_{n+1+i} or μ_h in (4.2) or (4.3) is zero or $\alpha_0 = (d-a(\gamma(1)) - \Sigma_{h=i}^{t_1+1} \lambda_h)/d$ in (4.3) with $\pi(i) = \gamma(1)$.

We show that $(\bar{p},\bar{y})$ is the endpoint of exactly one such line segment unless $(\bar{p},\bar{y})$ is an equilibrium. Because of assumption 4.3 there is exactly one index $i_0 \in I_{n+1+m}$ for which either $(z(\bar{p}) - A(\bar{y}))_{i_0} = \bar{\beta}$ if $i_0 \in I_{n+1}$ or $(A^T\bar{p})_{i_0-n-1} = \bar{\beta}$ if $i_0-n-1 \in I_m$ where $\bar{\beta} = \max\{\max_j(z(\bar{p}) - A\bar{y})_j, \max_i(A^T\bar{p})_i\}$.

Then $(\bar{p},\bar{y})$ is an equilibrium if $i_0 \in I_{n+1}$, $\bar{p} = e(i_0)$ and $\bar{y} = 0$. Otherwise the point $(\bar{p},\bar{y})$ is the endpoint of a line segment of points satisfying (4.1) in $\sigma((\bar{p},\bar{y}), (i_0), \emptyset)$ in $A((i_0), \emptyset)$ if $i_0 \in I_{n+1}$ and in the 1-cell $\sigma((\bar{p},\bar{y}), \{i_0\})$ in $A(\{i_0\})$ if $i_0 \in \{n+1,\ldots,n+m+1\}$.

In the first case the algorithm starts with a linear programming pivot step with λ_2 in (4.2) if $\bar{p}_{i_0} < 1$ and with α_0 in (4.3) if $\bar{p}_{i_0} = 1$. In the second case a linear programming pivot step is made with ν_{i_0} in (4.2).

In both cases $T = \{i_0\}$.

If a cell is complete then an approximating equilibrium is found.

Definition 4.5

A T-complete cell σ in A(T) is complete if at an endpoint (p^*,y^*) of the line segment of points satisfying (4.1) holds that

$$\bar{z}_j(p^*) - (Ay^*)_j < \beta^* \text{ implies } p^*_j = 0$$

and (4.8)

$$(A^Tp^*)_i < \beta^* \text{ implies } y^*_i = 0.$$

Theorem 4.6

Let ε be such that $\max_i(|z_i(p) - z_i(q)|) < \varepsilon$ for all p,q in the same simplex σ' of the V-triangulation of S^n with gridsize d^{-1}.

Let $\sigma((p^1,y^1), \pi(T_1),T_2)$ be a complete cell in A(T) with $(p^*,y^*) \in \sigma$ satisfying (4.8).

Then

$$|\beta^*| < \varepsilon(1+\Sigma y_i^*)^{-1}$$

$$\beta^* - \varepsilon < z_j(p^*) - (Ay^*)_j < \beta^* + \varepsilon \text{ if } p_j^* > 0$$

$$z_j(p^*) - (Ay^*)_j < \beta^* + \varepsilon \text{ if } p_j^* = 0.$$

Proof. Since according to (4.8) at (p^*,y^*)

$$\bar{z}_j(p^*) - (Ay^*)_j = \beta^* \text{ if } p_j^* > 0$$

and

$$(A^T p^*)_i = \beta^* \qquad \text{if } y_i^* > 0$$

we get

$$p^{*T}\bar{z}(p^*) - p^{*T}Ay^* = \beta^* \text{ and } p^{*T}Ay^* = \beta^* \sum_{i=1}^{m} y_i^*.$$

Hence $\beta^* = (p^{*T}\bar{z}(p^*))/(1+\sum_{i=1}^{m} y_i^*)$.

Furthermore,

$$|p^{*T}\bar{z}(p^*)| = |p^{*T}\bar{z}(p^*) - p^{*T}z(p^*)|$$

$$= |p^{*T}(\Sigma_j \lambda_j^* z(p^j) - \Sigma_j \lambda_j^* z(p^*))|$$

$$\leq p^{*T}\Sigma_j \lambda_j^* \|(z(p^j) - z(p^*))\|e < \varepsilon$$

with λ_j^*'s such that $p^* = \Sigma_j \lambda_j^* p^j$.

Therefore $|\beta^*| < \varepsilon (1+\sum_{i=1}^{m} y_i^*)^{-1}$.

Since $|\bar{z}_j(p^*) - z_j(p^*)| < \varepsilon$ for all $j \in I_{n+1}$ we obtain from (4.8) the rest of the theorem. □

Theorem 4.6 justifies that (p^*,y^*) is an approximate equilibrium in the sense that the inaccuracy ε can be made arbitrary small by choosing the gridsize d^{-1} small enough.

We will show now that an endpoint of a line segment of points satisfying (4.1) in a cell is either $(\bar{p},\bar{y})$ or an approximate solution or an endpoint of such a line segment in an adjacent cell. All this together yields that there is a piecewise linear path of points satisfying (4.1) which connects $(\bar{p},\bar{y})$ with an approximate equilibrium. This path will be traced by the algorithm through alternating replacement steps and linear programming pivot steps in (4.2) or (4.3).

More precisely we will show that an endpoint of a line segment of points satisfying (4.1) in a T-complete t-cell σ in A(T) is either $(\bar{p},\bar{y})$, an approximate equilibrium (p^*,y^*) or neither of both. In the last case either (p,y) is an endpoint of a line segment of points satisfying (4.1) in a $(T \cup \{h\})$-complete (t+1)-cell σ' in $A(T \cup \{h\})$ for some $h \notin T$ with σ a facet of σ', or (p,y) lies in a facet τ of σ. If τ lies in the boundary of A(T) then τ lies in $A(T\setminus\{h\})$ for some $h \in T$ and (p,y) is an endpoint of a line segment of points satisfying (4.1) in the $T\setminus\{h\}$ -complete (t-1)-cell τ in $A(T\setminus\{h\})$. Finally if τ does not lie in the boundary of A(T) then there is a unique t-cell σ' in A(T) containing τ also as a facet and (p,y) is an endpoint of a line segment of points satisfying (4.1) in the T-complete σ'.

Let $\sigma((p^1,y^1), \pi(T_1),T_2)$ be a T-complete t-cell in $A(\gamma(T_1),T_2)$ and $(p,y) \in \sigma$ an endpoint of a line segment of points satisfying (4.1). Because of the non-degeneracy assumption exactly one of the variables in (4.2) or (4.3) is equal to zero or α_0 is equal to its upperbound in (4.3). We subsequently discuss the cases that at (p,y) in system (4.2)

(i)	$\nu_{n+1+i} = 0$	for some $n+1+i \in T_2$
(ii)	$\mu_j = 0$	for some $j \notin T_1$, $j \in I_{n+1}$
(iii)	$\mu_{n+1+i} = 0$	for some $n+1+i \notin T_2$, $i \in I_m$
(iv)	$\lambda_j = 0$	for some $j \in I_{t_1+1}$

and in system (4.3)

(v)	$\nu_{n+1+i} = 0$	for some $n+1+i \in T_2$
(vi)	$\mu_j = 0$	for some $j \notin T_1$, $j \in I_{n+1}$
(vii)	$\mu_{n+1+i} = 0$	for some $n+1+i \notin T_2$, $i \in I_m$
(viii)	$\lambda_j = 0$	for some $j \in I_{t_1}$

(ix) $\lambda_0 = 0$

(x) $\alpha_0 = (d-a(\gamma(1)) - \sum_{h=1}^{t_1} \lambda_h)/d$ with $\pi(i) = \gamma(1)$.

(i) $v_{n+1+i} = 0$ for some $n+1+i \in T_2$.
If $T = \{n+1+i\}$ then $(p,y) = (\bar{p},\bar{y})$. Otherwise the endpoint (p,y) lies in a facet τ of σ where $v_{n+1+i} = 0$. Therefore (p,y) is also an endpoint of a line segment (of points satisfying (4.1)) in τ which is the $(t-1)$-cell $\sigma((p^1,y^1), \pi(T_1),T_2\setminus\{n+1+i\})$ in $A(\gamma(T_1),T_2\setminus\{n+1+i\})$. This line segment can be followed by making a pivot step in (4.2) with μ_{n+1+i}.

(ii) $\mu_j = 0$ for some $j \notin T_1$, $j \in I_{n+1}$.

(a) If $T_1 \cup \{j\} = I_{n+1}$ or $\bar{p}_k = 0$ for all $k \notin T_1 \cup \{j\}$ and if $\bar{y}_i = 0$ for all $n+1+i \notin T_2$ then according to 4.2 σ is complete and (p,y) is an approximate equilibrium.

(b) If there exists a k not in $T_1 \cup \{j\}$ with $\bar{p}_k > 0$ then (p,y) is an endpoint of a line segment in the $(T \cup \{j\})$-complete $(t+1)$-cell $\sigma((p^1,y^1), (\pi(T_1),j),T_2)$ in $A((\gamma(T_1),j),T_2)$. This line segment can be followed by making a pivot step in (4.2) with λ_{t_1+2}.

(c) If $T_1 \cup \{j\} = I_{n+1}$ or $\bar{p}_k = 0$ for all $k \notin T_1 \cup \{j\}$ and if there is an $n+1+i \notin T_2$ such that $\bar{y}_i > 0$, then (p,y) is an endpoint of a line segment in the $(T \cup \{j\})$-complete $(t+1)$-cell $\sigma((p^1,y^1),(\pi(T_1),j),T_2)$ in $A((\gamma(T_1),j),T_2)$. This line segment can be followed by making a pivot step in (4.3) with α_0.

(iii) $\mu_{n+1+i} = 0$ for some $n+1+i \notin T_2$, $i \in I_m$.
The point (p,y) is an endpoint of a line segment in the $(T \cup \{n+1+i\})$-complete $(t+1)$-cell $\sigma((p^1,y^1),\pi(T_1), T_2 \cup \{n+1+i\})$ in $A(\gamma(T_1),T_2 \cup \{n+1+i\})$.
This line segment can be followed by making a pivot step in (4.2) with v_{n+1+i}.

(iv) $\lambda_j = 0$ for some $j \leq t_1+1$.

(a) If $j=1$, $\pi(1) = \gamma(1)$ and $a(\pi(1)) = d-1$ then $(p,y) \in$ Bd $A(T)$ where $p_j = 0$ for all $j \notin T_1$ and $y_i = 0$ for all $n+1+i \notin T_2$. According to 4.2 σ is complete and (p,y) is an approximate equilibrium.

(b) If $j = t_1+1$, $\pi(t_1) = \gamma(t_1)$ and $a(\pi(t_1)) = 0$ then $(p,y) = (\bar{p},\bar{y})$ when $t_1 = 1$ and $T_2 = \emptyset$ and otherwise $(p,y) \in$ Bd $A(T)$ where $\alpha_{\pi(t_1)} = 0$. In this case (p,y) is an endpoint of a line segment in the $(T\setminus\{\pi(t_1)\})$-complete $(t-1)$-cell $\sigma((p^1,y^1), (\pi(1),\ldots,\pi(t_1-1)),T_2)$ in $A((\gamma(1),\ldots,\gamma(t_1-1)),T_2)$.

This line segment can be followed by making a pivot step in (4.2) with $\mu_{\pi(t_1)}$.

(c) In the other cases (p,y) is an endpoint of a line segment in an adjacent T-complete t-cell ρ in $A(T)$. More precisely, if for some i, $2 \leq i \leq t_1$, $\pi(j) = \gamma(i)$, $\pi(j-1) = \gamma(i-1)$ and $a(\pi(j)) = a(\pi(j-1))$ then according to (3.4) ρ is the t-cell $\sigma((p^1,y^1),\pi'(T_1),T_2)$ in $A(\gamma'(T_1),T_2)$ with $\gamma'(T_1) = (\gamma(1),\ldots,\gamma(i-2), \gamma(i), \gamma(i-1), \gamma(i+1),\ldots,\gamma(t_1))$ and $\pi'(T_1) = (\pi(1),\ldots,\pi(j-2), \pi(j), \pi(j-1), \pi(j+1),\ldots,\pi(t_1))$. Otherwise ρ is the t-cell $\sigma((p^1,y^1), \pi(T_1),T_2)$ in $A(\gamma(T_1),T_2)$ obtained from σ according to the replacement rule described in table 4.1 where $a_h = a(h)$ if $h \in T_1$ and $a_h = 0$ if $h \in I_{n+1}\setminus T_1$. The line segment in ρ from (p,y) can be followed by making a pivot step in (4.2) with λ_h where h is the index of the new vertex of ρ.

	(p^1,y^1) becomes	$\pi(T_1)$ becomes	a becomes
j=1	$(p^1,y^1)+d^{-1}q(\pi(1))$	$(\pi(2),\ldots,\pi(t_1),\pi(1))$	$a+e_1(\pi(1))$
$1 < j < t_1+1$	(p^1,y^1)	$(\pi(1),\ldots,\pi(j-2),\pi(j),$ $\pi(j-1),\pi(j+1),\ldots,\pi(t_1))$	a
$j = t_1+1$	$(p^1,y^1)-d^{-1}q(\pi(t_1))$	$(\pi(t_1),\pi(1),\ldots,\pi(t_1-1))$	$a-e_1(\pi(t_1))$

Table 4.1. Replacement rule

Now we describe the cases for system (4.3).

(v) $\nu_{n+1+i} = 0$ for some $n+1+i \in T_2$.
Idem to case (i) except that a pivot step is made in (4.3) with μ_{n+1+i}.

(vi) $\mu_j = 0$ for some $j \notin T_1$, $j \in I_{n+1}$.
Remark that $\bar{p}_j = 0$. Then according to (3.5) (p,y) is an endpoint of a line segment in the $(T \cup \{j\})$-complete (t+1)-cell $\sigma((p^1,y^1),(\pi(1),\ldots,\pi(t_1-1),j,\pi(t_1)),T_2)$ in $A((\gamma(1),\ldots,\gamma(t_1-1),j,\gamma(t_1)),T_2)$. This line segment can be followed by making a pivot step in (4.3) with λ_{t_1+1}.

(vii) $\mu_{n+1+i} = 0$ for some $n+1+i \notin T_2$, $i \in I_m$.
(a) If $\bar{y}_h = 0$ for all $n+1+h \notin (T_2 \cup \{n+1+i\})$ then also $y_h = 0$ for these h's, so that σ is complete and (p,y) is an approximate equilibrium.
(b) Otherwise idem to (iii) except that a pivot step is made in (4.3) with ν_{n+1+i}.

(viii) $\lambda_j = 0$ for some $j \in I_{t_1}$.
(a) If j=1, $\pi(1) = \gamma(1)$ and $a(\pi(1)) = d-1$ then idem to (iv) (a).
(b) If $j = t_1$, $\pi(t_1-1) = \gamma(t_1-1)$, $a(\pi(t_1-1)) = 0$ and $\bar{p}_{\gamma(t_1-1)} = 0$

then (p,y) is an endpoint of a line segment in the $(T\setminus\{\pi(t_1-1)\})$-complete $(t-1)$-cell $\sigma((p^1,y^1),(\pi(1),\ldots,\pi(t_1-2),\pi(t_1)),T_2)$ in $A((\gamma(1),\ldots,\gamma(t_1-2),\gamma(t_1)),T_2)$.
This line segment can be followed by making a pivot step in (4.3) with μ_h where $h = \pi(t_1-1)$.

(c) In the other cases idem to (iv) (c), except that the replacement rule is performed as described in table 4.2 and that a pivot step is made in (4.3).

	(p^1,y^1) becomes	$\pi(T_1)$ becomes	a becomes
$j=1$	$(p^1,y^1)+d^{-1}q(\pi(1))$	$(\pi(2),\ldots,\pi(t_1-1),\pi(1),\pi(t_1))$	$a+e_1(\pi(1))$
$1<j<t_1$	(p^1,y^1)	$(\pi(1),\ldots,\pi(j-2),\pi(j),\pi(j-1),\pi(j+1),\ldots,\pi(t_1))$	a
$j=t_1$	$(p^1,y^1)-d^{-1}q(\pi(t_1-1))$	$(\pi(t_1-1),\pi(1),\ldots,\pi(t_1-2),\pi(t_1))$	$a-e_1(\pi(t_1-1))$

Table 4.2. Replacement rule

(ix) $\alpha_0 = 0$.
The point (p,y) is the starting point $(\bar{p},\bar{y})$ if $t=1$ and otherwise (p,y) is an endpoint of a line segment in the $(T\setminus\{\gamma(t_1)\})$-complete $(t-1)$-cell $\sigma((p^1,y^1),(\pi(1),\ldots,\pi(t_1-1)),T_2)$ in $A((\gamma(1),\ldots,\gamma(t_1-1)),T_2)$. This line segment can be followed by making a pivot step in (4.2) with $\mu_{\gamma(t_1)}$.

(x) $\alpha_0 = (d-a(\gamma(1)) - \sum_{h=i}^{t_1} \lambda_h)/d$ with $\pi(i) = \gamma(1)$.
According to (3.5) $y_i = 0$ for all $n+1+i \notin T_2$ so that σ is complete and (p,y) is an approximate equilibrium.

This completes the description of the movements from a line segment of points satisfying (4.1) in one cell to the line segment of such points in an adjacent cell. The algorithm follows the sequence of adjacent line

segments from $(\bar{p},\bar{y})$ to an approximate equilibrium. Since no cell can be visited more than once, the number of cells is finite and because of theorem 4.4, the algorithm must terminate within a finite number of steps with an approximate solution (p^*,y^*) in a complete cell.

When the accuracy of the approximation at (p^*,y^*) is not sufficient the algorithm can be restarted at (p^*,y^*) with a smaller grid size d^{-1}. Within a finite number of restarts any (a priori) accuracy of approximation can be reached.

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